Taylor series Cheat Sheet

Edexcel FP1

Taylor series are used to approximate functions. Approximating a more complicated function by an infinite sum of polynomials means it can be solved numerically. This means that Taylor series have lots of applications in physics and engineering. Taylor series also allows integrals of functions with no antiderivative to be approximated, but in this topic we will focus on using Taylor series to find approximate solutions to differential equations that can't be solved easily by other methods.

Taylor series

In the second book for Core Pure, Maclaurin series were introduced. As a recap, Maclaurin series allow a function of x that is infinitely differentiable, with the derivatives defined for all $n \in \mathbb{N}$, to be written as an infinite series in ascending powers of x, and focuses on $x = 0$. Clearly, this is not ideal, as not all functions have derivatives that are defined for all natural numbers, such as $\ln x$. To overcome this, we derive a series expansion that focuses on $x = a$ instead, which we call Taylor series and is a more general form of the Maclaurin series, which is given in two different forms:

- $f(x+a) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \frac{f'''(a)}{3!}x^3 + \cdots + \frac{f^{(r)}(a)}{r!}x^r + \cdots$
- $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(r)}(a)}{r!}(x-a)^r + \cdots$

These expansions are known as Taylor series of $f(x)$ at the point $x = a$. The Taylor series expansion is only valid if $f^{(n)}(a)$ exists and is finite for all $n \in \mathbb{N}$, and for values of x for which the infinite series converges. The Taylor series expansion is **not** given in the formula booklet- it is essential that you learn them both!

Example 1: Find the Taylor series of $\sin x$ about the point $x = \frac{\pi}{3}$ up to and including the term x^3.

Use the first expansion given: $f(x+a) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \frac{f'''(a)}{3!}x^3 + \cdots + \frac{f^{(r)}(a)}{r!}x^r + \cdots$	If $f(x) = \sin x$, then $\sin\left(x + \frac{\pi}{3}\right) = f(x + \frac{\pi}{3})$
Find the first, second and third derivatives at the point a, since we only need to expand the series up to the term x^3.	$f(x) = \sin x \Longrightarrow f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ $f'(x) = \cos x \Rightarrow f'\left(\frac{\pi}{3}\right) = \frac{1}{2}$ $f''(x) = -\sin x \Longrightarrow f''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ $f'''(x) = -\cos x \Rightarrow f'''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$
Substitute the values found into the expansion.	$\sin\left(x + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}x - \left(\frac{\sqrt{3}}{2} \times \frac{1}{2!}\right)x^2 - \left(\frac{1}{2} \times \frac{1}{3!}\right)x^3 + \cdots$
Simplify the expression	$\sin\left(x + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}x - \frac{\sqrt{3}}{4}x^2 - \frac{1}{12}x^3 + \cdots$

Finding limits

Previously, you have considered limits of a function as x approaches 0 or ∞ by looking at how different parts of the functions behave. It is also possible to evaluate limits of a function as x approaches a certain value a, which is denoted $\lim_{x \to a} f(x) = L$, where L is the numerical value of the limit. Limits can be found in many different ways, with the simplest way being to separate the limit into other limits that you already know, and follow properties that are sometimes referred to as the algebra of limits:

- Given $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then:
- $\lim_{x \to a}\big(f(x) + g(x)\big) = L + M$
- For a constant c, $\lim_{x \to a} c\, f(x) = cL$
- $\lim_{x \to a} f(x)g(x) = LM$
- If $M \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$

Example 2: Evaluate the limit $\lim_{x \to \infty} \frac{5-3x}{4+x}$

We can begin by trying to use the rule 'if $M \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$'	$\lim_{x \to \infty} 5 - 3x = -\infty$ $\lim_{x \to \infty} 4 + x = \infty$ This leaves the original limit as $-\frac{\infty}{\infty}$, which is indeterminate, and we cannot determine the limit directly. This means we need to change the form of the limit.
Divide the numerator and denominator by x.	$\lim_{x \to \infty} \frac{5-3x}{4+x} = \lim_{x \to \infty} \frac{\frac{5}{x} - 3}{\frac{4}{x} + 1}$
Apply the rule 'if $M \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$'	$\lim_{x \to \infty} \frac{5}{x} - 3 = -3$ $\lim_{x \to \infty} \frac{4}{x} + 1 = 1$ So, $\lim_{x \to \infty} \frac{5-3x}{4+x} = \lim_{x \to \infty} \frac{\frac{5}{x}-3}{\frac{4}{x}+1} = -\frac{3}{1} = -3$

Clearly, we can't always just substitute the value that x tends to into the limit and evaluate- consider the limit $\lim_{x \to 0} \frac{\sin x}{x}$. Like the limit in example 2, it is indeterminate as by substituting in $x = 0$ we get $\frac{0}{0}$. To evaluate this type of limit we need a more precise method- we can use the Taylor series at $x = 0$ (otherwise known as the Maclaurin series) to do this:

Example 3: Using Taylor series, evaluate the limit $\lim_{x \to 0} \frac{\sin x}{x}$

Find the Taylor series for $\sin x$ about $x = 0$ (this is also the Maclaurin series)	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
Divide by x	$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$
Evaluate the limit using the series- all of the terms involving x will tend to 0	$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0}(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots) = 1$

This method can also be used to evaluate more complex limits

Example 4: Evaluate the limit $\lim_{x \to 0} \frac{\ln(1+x^2)}{x^2}$

Find the Taylor expansion for $\ln(1+x^2)$	$f(x) = \ln x^2 \Rightarrow f(x+1) = \ln(x^2+1)$ $f(x) = \ln x^2 \Rightarrow f(a) = 0$ $f'(x) = \frac{2}{x} \Rightarrow f'(a) = 2$ $f''(x) = \frac{-2}{x^2} \Rightarrow f''(a) = -2$ $f'''(x) = \frac{4}{x^3} \Rightarrow f'''(a) = 4$ Taylor expansion: $\ln(x^2+1) = 2x - \frac{2}{2!}x^2 + \frac{4}{3!}x^3 + \cdots$ $\ln(x^2+1) = 2x - x^2 + \frac{2}{3}x^3 + \cdots$
Substitute the Taylors expansion into the limit	$\lim_{x \to 0} \frac{\ln(1+x^2)}{x^2} = \lim_{x \to 0} \frac{2x - x^2 + \frac{2}{3}x^3}{x^2}$ $= \lim_{x \to 0} \frac{2}{x} - 1 + \frac{2}{3}x$ $= \frac{2}{0} = \infty$

Series solutions of differential equations

Taylor series can be used to approximate solutions of differential equations that can't be solved using other techniques. These approximate solutions are in the form of series, and are hence called series solutions.

- The series solution to the differential equation $\frac{dy}{dx} = f(x,y)$ is found using the Taylor series expansion in the form:
$$y = y_0 + (x - x_0)\frac{dy}{dx}\Big|_{x_0} + \frac{(x-x_0)^2}{2!}\frac{d^2y}{dx^2}\Big|_{x_0} + \frac{(x-x_0)^3}{3!}\frac{d^3y}{dx^3}\Big|_{x_0} + \cdots$$
- When $x_0 = 0$, this reduces to the Maclaurin series
$$y = y_0 + x\frac{dy}{dx}\Big|_0 + \frac{x^2}{2!}\frac{d^2y}{dx^2}\Big|_0 + \frac{x^3}{3!}\frac{d^3y}{dx^3}\Big|_0 + \cdots$$

As we have the first order differential of the form $\frac{dy}{dx} = f(x,y)$ with initial conditions, we can calculate $\frac{dy}{dx}\big|_{x_0}$ by substituting these initial conditions in. If we differentiate the original equation, we can obtain $\frac{d^2y}{dx^2}$, and thus find the value at the initial conditions by substituting them in. Repeated differentiation and substitution allows us to find higher derivatives.

Example 5: Use the Taylor series method to find a series solution, in ascending powers of $(x-1)$ up to and including $(x-1)^2$ of $\frac{dy}{dx} = e^{xy} + x^3$, given that when $x = 1, y = 2$.

Substitute the given conditions into the original equation	The given conditions are $x_0 = 1, y_0 = 2$, so $\frac{dy}{dx}\big|_{x_0} = e^2 + 1$
Differentiate the original equation then substitute the given conditions in- remember to differentiate implicitly	$\frac{d^2y}{dx^2} = ye^{xy} + x\frac{dy}{dx}e^{xy} + 3x^2$ Substituting in $x_0 = 1, y_0 = 2, \frac{dy}{dx}\big|_{x_0} = e^2 + 1$ $\frac{d^2y}{dx^2} = 2e^2 + (e^2+1)e^2 + 3$
Substitute into the Taylor series expansion formula- we cannot use the Maclaurin series one as $x_0 \neq 0$.	$y = 2 + (x-1)(e^2+1) + \frac{(x-1)^2}{2}(2e^2 + (e^2+1)e^2 + 3) + \cdots$

Second, and higher order differential equations can be solved in the same manner as you will be given extra initial conditions:

Example 5: Use the Taylor series method to find a series solution, in ascending powers of x up to and including the term x^3, of $\frac{d^2y}{dx^2} = y^2 - \cos 2x$, with initial conditions $x = 0, y = 2\ and\ \frac{dy}{dx} = 1$

Substitute the initial conditions into the original equations	The given conditions are $x = 0, y = 2\ and\ \frac{dy}{dx} = 1$, so $\frac{d^2y}{dx^2} = 4 - 1 = 3$
Differentiate the original equation implicitly and substitute the initial conditions in	$\frac{d^3y}{dx^3} = 2y\frac{dy}{dx} + 2\sin 2x$ $\frac{d^3y}{dx^3} = 2(2)(1) + 2(0) = 4$
Substitute into the Maclaurin expansion	$y = 2 + x + \frac{3}{2}x^2 + \frac{4}{3!}x^3 + \cdots$ $y = 2 + x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \cdots$

Example 6 (Mixed Exercise): Write down the Taylor expansion of $\sin 3x$ and $\cos 3x$ about the point $\frac{\pi}{4}$ up to and including powers of x^4.
Hence, or otherwise, write the Taylor expansion of $\tan 3x$ about the point $\frac{\pi}{4}$ up to and including powers of x^4.

Use the expansion $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(r)}(a)}{r!}(x-a)^r + \cdots$	If $f(x) = \sin 3x$, then the point of expansion $a = \frac{\pi}{4}$
Find the first, second, third and fourth derivatives at a	$f(x) = \sin 3x \Rightarrow f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $f'(x) = 3\cos 3x \Rightarrow f'\left(\frac{\pi}{4}\right) = -\frac{3\sqrt{2}}{2}$ $f''(x) = -9\sin 3x \Rightarrow f''\left(\frac{\pi}{4}\right) = \frac{-9\sqrt{2}}{2}$ $f'''(x) = -27\cos 3x \Rightarrow f'''\left(\frac{\pi}{4}\right) = \frac{27\sqrt{2}}{2}$ $f''''(x) = 81\sin 3x \Rightarrow f'''''\left(\frac{\pi}{4}\right) = \frac{81\sqrt{2}}{2}$
Substitute into the expansion- remember the factorials on the denominator	$\sin 3x = \frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{9\sqrt{2}}{4}(x - \frac{\pi}{4})^2 + \frac{9\sqrt{2}}{4}(x - \frac{\pi}{4})^3 + \frac{27\sqrt{2}}{16}(x - \frac{\pi}{4})^4 + \cdots$
Repeat the process with $\cos 3x$	$f(x) = \cos 3x \Rightarrow f\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ $f'(x) = -3\sin 3x \Rightarrow f'\left(\frac{\pi}{4}\right) = \frac{-3\sqrt{2}}{2}$ $f''(x) = -9\cos 3x \Rightarrow f''\left(\frac{\pi}{4}\right) = \frac{9\sqrt{2}}{2}$ $f'''(x) = 27\sin 3x \Rightarrow f'''\left(\frac{\pi}{4}\right) = \frac{27\sqrt{2}}{2}$ $f''''(x) = 81\cos 3x \Rightarrow f'''''\left(\frac{\pi}{4}\right) = -\frac{81\sqrt{2}}{2}$
Substitute into the expansion	$\cos 3x = -\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{9\sqrt{2}}{4}(x - \frac{\pi}{4})^2 + \frac{9\sqrt{2}}{4}(x - \frac{\pi}{4})^3 - \frac{27\sqrt{2}}{16}(x - \frac{\pi}{4})^4 + \cdots$
We can use the formula $\tan x = \frac{\sin x}{\cos x}$ to find the expansion for $\tan 3x$	$\tan 3x = \frac{\sin 3x}{\cos 3x}$ $\tan 3x = (\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{9\sqrt{2}}{4}(x - \frac{\pi}{4})^2 + \frac{9\sqrt{2}}{4}(x - \frac{\pi}{4})^3 + \frac{27\sqrt{2}}{16}(x - \frac{\pi}{4})^4)$ $\div -\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{9\sqrt{2}}{4}(x - \frac{\pi}{4})^2 + \frac{9\sqrt{2}}{4}(x - \frac{\pi}{4})^3 - \frac{27\sqrt{2}}{16}(x - \frac{\pi}{4})^4$
Using long division we obtain	$\tan 3x = -1 + 6\left(x - \frac{\pi}{4}\right) - 18(x - \frac{\pi}{4})^2 + 72(x - \frac{\pi}{4})^3 - 270(x - \frac{\pi}{4})^4 + \cdots$